September 23, 2011 **Solution**
Analysis I - Mid-Sem Exam - Semester I

1. *Let (a_n) and (b_n) be sequences converging to a and b respectively. Prove that (i) $a_n + b_n \to a + b$ and $ra_n \to ra$, (ii) (a_n) is a bounded sequence.*

   Solution: (i) For a given $\epsilon > 0$, there exists a natural number N such that $|a_n - a| < \epsilon/2$ and $|b_n - b| < \epsilon/2$ for all $n \geq N$. Thus, $|a_n + b_n - (a + b)| \leq \epsilon/2 + \epsilon/2 = \epsilon$ for $n \geq N$. Hence $a_n + b_n$ goes to $a + b$. Similiarly, there exists K, natural number, such that $|a_n - a| \leq \epsilon/r$ for $n \geq K$; (the case when $r = 0$ is trivial, so here we have assumed r is nonzero.) Hence $|ra_n - ra| < \epsilon$.

   (ii) To show the bounedness of a convergent sequence (a_n), we need to find an $M > 0$ such that $|a_n| \leq M$ for all n. We assume that $a \geq 0$; (the other case can be proved as $-a_n$ converges to $-a$.) For $\epsilon = 2a + 2$, we find $N \in \mathbb{N}$ such that $|a_n - a| \leq \epsilon$, ie, $-1 < a_n < 3a + 1$ for all $n \geq N$. Hence $|a_n| < \max\{1, 3a + 1\}$ for all $n \geq N$. Taking alongwith the maximum of $|a_1|, ..., |a_{N-1}|$, we get the boundedness. □

2. *Let (x_n) and (y_n) be bounded sequences. Then show that $\underline{lim}(-x_n) = -\overline{lim}(x_n)$, and $\underline{lim}(x_n) + \underline{lim}(y_n) \leq \underline{lim}(x_n + y_n) \leq \overline{lim}(x_n) + \underline{lim}(y_n) \leq \overline{lim}(x_n + y_n) \leq \overline{lim}(x_n) + \overline{lim}(y_n)$.*

   Solution: We have that $\underline{lim}(x_n) = \lim_{k\to\infty} \inf_{n\geq k} x_n$, and $\overline{lim}(x_n) = \lim_{k\to\infty} \sup_{n\geq k} x_n$.

   Now, $\underline{lim}(-x_n) = \lim_{k\to\infty}\ inf_{n\geq k}(-x_n) = \lim_k(-\sup_{n\geq k} x_k)$ gives the first part.

   For the part second, we have the following, assuming that the sequences are bounded.

   $\inf_{j\geq k} x_j \leq x_j$ and $\inf_{j\geq k} y_j \leq y_j$ for each $j \geq k$ implies $\inf_{j\geq k} x_j + \inf_{j\geq k} y_j \leq x_j + y_j$ for $j \geq k$. Hence $\lim_{k\to\infty} \inf_{j\geq k} x_j + \lim_{k\to\infty} \inf_{j\geq k} y_j \leq \lim_{k\to\infty} \inf_{j\geq k}(x_j + y_j)$. This gives the first inequality.

   $x_j \leq \sup_{j\geq k} x_j$ for all $j \geq k$ implies $x_j + y_j \leq \sup_{j\geq k} x_j + y_j$. Hence $\inf_{j\geq k}(x_j + y_j) \leq \sup_{j\geq k} + \inf_{j\geq k} y_j$. The second inequaltiy follows.

   Similarly, we can prove the other inequalities.

   □

3. (i) Show that if r is the limit point of a sequence (a_n), then there exists a subsequence (a_{n_k}) converging to r. (ii) If (a_n) is defined by

   $$a_0 = 0, a_{2m} = \tfrac{a_{2m-1}}{2} \quad and \quad a_{2m+1} = \tfrac{1}{2} + a_{2m},$$

   then find lim $\inf_n a_n$ and lim $\sup_n a_n$.

   Solution: (i) Consider the interval $I_k = (r - 1/k, r + 1/k$, where $k \geq 1$. By defintion, each of these intervals contains terms from the given sequence. We can choose (a_{n_k}) in such away that $a_{n_k} \in I_k$, but not in I_{k+1}; in this way we get subsequence with different terms. The result follows since, for a given $\epsilon > 0$, one can find N such that $1/k < \epsilon$ for all $k \geq N$ and also $|a_{n_k} - r| < 1/k$ for all k.

   (ii) Observe that the sequence is

$$0, 0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2} + \frac{1}{2^2}, \frac{1}{2^2} + \frac{1}{2^3}, \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} \cdots$$

Each term is positive. We guess that the lim sup and lim inf are respectively 1 and 1/2 respectively as the the above sequence can be split into two subsequences by collecting all the odd and even terms separately; one of them converges to 1/2 and other converges to 1. Infimum and supremum are $1/2$ and 1.

□

4. Prove that (i)$lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty}(n-1)^{1/n} = 1$, (ii) every Cauchy sequence converges.

   Solution: (i) Let $y_n = n^{1/n}$. So, $\log y_n = \log n/n$, $n \geq 1$. Using L' Hopital rule for $\log x/x$, we get $\log y_n$ converges to 0. Hence y_n converges to 1. The same proof applies to the other limit in the question.

   (ii) Let (a_n) be a Cauchy sequence of real numbers. First, observe that (a_n) is bounded which is exactly similar to the proof of part (ii) in the question 1; (indeed, there exists a natural number N such that $|a_n - a_N| < 1$ for all $n \geq N$. Now proceed as in the first question.) Now we apply the Bolzano-Weierstrass theorem, namely every bounded real sequence has a convergent subsequence, to (a_n). Let $(a_{n_k})_k$ be a subsequence converging to a. We claim that (a_n) itself converges to a. By definition, for a given $\epsilon > 0$, there exists $N_1 \in \mathbb{N}_1$ such that $|a_{n_k} - a| < \epsilon/2$ for all $k \geq N_1$. By Cauchy property, we also have $|a_n - a_m| < \epsilon/2$ for all $n, m \geq N_2$ for some N_2. Take N to be any number in (n_k) , that is more than the maximum of N_1 and N_2. Hence, for $j \geq N$, we have $|a_j - a| \leq |a_j - a_N| + |a_N - a$. This gives the result. □

5. Prove that (i) If $|a_n| \leq c_n$ for all n and the series $\sum c_n$ converges, then the series $\sum a_n$ converges. (ii) Let (a_i) be a decreasing sequence of non-negative numbers. Prove that $\sum a_n$ converges if and only of $\sum 2^n a_{2^n}$ converges.

   Solution: (i) By the comparison test, the series $\sum a_n$ converges absolutely. It is known that aboslute convergence implies convergence in $\mathbb{R}$.

   (ii) Both the series terms are positive. Hence, to use the comparison test, it suffices to prove that if the partial sum of one series is bounded then that of the series is bounded. For this, let $s_n = \sum_{k=1}^{n} a_k$ and $S_n = \sum_{k=0}^{n-1} 2^k a_{2^k}$ be respectively the n-th partial sums of the two series. Then

   $$s_n \leq s_{2^n - 1} \leq S_n \text{ since } a_k \leq a_{2^{m-1}} \text{ for } 2^{m-1} \leq k < 2^m.$$

   Also $S_n < 2s_{2^{n-1}}$ since

   $$2^k a_{2^k} \leq 2 \sum_{m=2^{k-1}+1}^{2^k} a_m$$

   □

6. Prove that (i) If the sequence of partial sums of $\sum_n a_n$ is bounded and (b_n) is either monotonically increasing or decreasing to 0, then $\sum a_n b_n$ converges. (ii) $\sum_n a_n$ converges and b_n is a bounded monotonic sequence, then $\sum_n a_n b_n$ converges.

   Solution: (i) Claim: the partial sum sequence of $\sum_n a_n b_n$ is Cauchy. Assume that (b_n) is decreasing to 0. Split the sum $\sum_{n=j}^{k} a_j b_j$ as two sums, so that in the first we have

all a_n terms neagative and in the second sum, all a_n positive. From the positive sum part, take out the maximum of b_j's and in the negative part, take out the minimum of $-b_j$. Use the hypotheses. The proof of (ii) is similar.

□

7. (i) Find the readius of convergence of the following series:

$$\frac{1}{3} + \frac{1}{5}z + \frac{1}{3^2}z^2 + \frac{1}{5^2}z^3 + \frac{1}{3^3}z^4 + \frac{1}{5^3}z^5 + ...$$

(ii) Let (a_n) be a sequence. Put $p_n = |a_n| + a_n$ and $q_n = |a_n| - a_n$. Prove the following. (a) $\sum p_n$ and $\sum q_n$ converge if and only if $\sum a_n$ converges absolutely. (b) If $\sum a_n$ and $\sum p_n$ converge, then $\sum a_n$ converges absolutely.

Solution: (i) The radius of convergence R is given by $R^{-1} = \limsup |a_n|^{1/n}$ for a series $\sum a_n z^n$. In this case, we find it separately for series with all odd terms and all even terms. Then,

$$\limsup |a_{2n}|^{1/2n} = \lim \frac{1}{3^{(n+1)/2n}} \textbf{ and } \limsup |a_{2n+1})|^{1/(2n+1)} = \lim \frac{1}{5^{(n+1)/2n+1}}$$

These values are $\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{5}}$ respectively. Hence, the radius of convergence is $\sqrt{3}$.

(a) These follow reasily, as p_n and q_n are non-negative, and $p_n + q_n = 2|a_n|$. So, if $\sum p_n$ and $\sum q_n$ converge, so does $\sum |a_n|$. Conversly if this series converges absolutely, then the observation $\sum p_n \leq 2 \sum |a_n|$ alongwith the comparison gives the result. (b) Observe $|a_n| = p_n - a_n$, hence the result. □